ndeed, $(uv)^n$ can be obtained by expanding $(u+v)^n$ using binomial theorem where the powers are interpreted as derivatives. So the nth derivative expressions

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)}$$

Example 1: Given $y = x^3 e^{4x}$, use the leibnitz theorem to determine the values of: (i) y^n, (ii) $y^{(5)}$ (iii) $y^{(8)}$

Solution

$u = e^{4x} \Rightarrow u' = 4e^{4x} \Rightarrow u'' = 4^2 e^{4x} \Rightarrow u''' = 4^3 e^{4x} \Rightarrow u^n = 4^n e^{4x}$

$u^{n-1} = 4^{n-1} e^{4x}$.

$v = x^3, \Rightarrow v' = 3x^2, \; v'' = 6x; \; v''' = 6; \; v^{(4)} = 0$

from the leibnitz theorem

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)}$$

(i) $$y^n = 4^n e^{4x} x^3 + n.4^{n-1}e^{4x}.3x^2 + \frac{n(n-1)}{2!}4^{n-2}e^{4x}.6x + \frac{n(n-1)(n-2)}{3!}4^{n-3}e^{4x}.6$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}.4^{n-3}e^{4x}.0$$

$$y^n = 4^n e^{4x}x^3 + \frac{3n\,4^n e^{4x}x^2}{4} + \frac{3n(n-1)4^n e^{4x}}{16} + \frac{n(n-1)(n-2)4^n e^{4x}}{64} + 0$$

multiply through by 64

$$64y^n = 4^n e^{4x}\{64x^3 + 48nx^2 + 12n(n-1)x + n(n-1)(n-2)\}$$

(i) $$y^n = 4^{n-3}e^{4x}\{64x^3 + 48nx^2 + 12n(n-1)x + n(n-1)(n-2)\}$$ Ans

Sequence

A Sequence is a set of quantities u_1, u_2, u_3 ---, stated in a definite order and each term formed according to a fixed pattern

i.e $u_r = f(r)$

eg - $1, 3, 5, 7, \ldots$ is a sequence (the next term = 9)

$2, 6, 18, 54 \ldots$ ✓ ✓ ✓ 162

$1, -2^2, 3^2, -4^2 \ldots$ ✓ ✓ ✓ ✓ $+5^2$

$1, -5, 37, 6 \ldots$ is a sequence, but its pattern is more involved and the next term cannot readily be anticipated.

A finite sequence contains only a finite number of terms. An infinite sequence is unending.

Series

A Series is formed by the sum of the terms of a sequence.

eg $1, 3, 5, 7 \ldots$ is a sequence

but $1+3+5+7 \ldots$ is a series.

Arithmetic Series (or arithmetic Progression) AP

eg $2 + 5 + 8 + 11 + 14 + \ldots$

You will notice that each term can be written from ~~the~~ previous term by adding on a constant value 3.

Suppose we wish to express $\sin x$ as a series of ascending powers of x. The series will be of the form

$$\sin x \equiv a + bx + cx^2 + dx^3 + ex^4 + \ldots$$

to establish the series, we have to find the values of the constant coefficients a, b, c, d, etc.

If $x = 0$, on both sides,

$$\sin 0 = a + 0 + \infty =, \; a = 0$$

If we could substitute some other value of x, which will make all the terms disappear except the second we should then find the value of 'b'. Unfortunately there is no such substitute. Hence the key is to Differentiate both sides with respect to x

Thus; we have

$$\cos x = b + c2x + d3x^2 + e4x^3 + \ldots$$

Note; this is still an Identity, so we can substitute to it any value for 'x' we like.

Notice that the 'a' has disappeared and the constant term at the beginning is now 'b'.

If we substitute $x = 0$

$\cos 0 = 1 = b + 0 + 0 + 0 = 1$, $b = 1$

to find ~~c~~ and d, we repeat ~~the~~ process.

Finally, $\sin x = 0 + 1.x + 0.x^2 + -\frac{1}{3!}x^3 + 0.x^4 + \frac{1}{5!}x^5 +$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

This method would enable us to express any function as a series of ascending powers of x, as long as we can differentiate a given function over and over again; and find the values of the derivatives when we put $x = 0$.

However it entails a considerable amount of writing, thus a general form of such series is established called Maclaurin's Series.

Maclaurins Series :

Working with a general function $f(x)$, instead of $\sin x$

Let $f(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$

Put $x=0$, then $f(0) = a + 0 + 0 + \ldots$, $a = f(0)$

Differentiate

$$f'(x) = b + c\cdot 2x + d\cdot 3x^2 + e\cdot 4x^3 + f\cdot 5x^4$$

Put $x=0$, $f'(0) = b + 0 + 0$ $\therefore$ $b = f'(0)$

Differentiate

$$f''(x) = c\cdot 2\cdot 1 + d\cdot 3\cdot 2x + e\cdot 4\cdot 3x^2 + f5\cdot 4x^2$$

Put $x=0$, $f''(0) = c\cdot 2! + 0 + 0 +$; $c = \dfrac{f''(0)}{2!}$

Differentiate

$$f'''(x) = d\cdot 3\cdot 2\cdot 1 + e\cdot 4\cdot 3\cdot 2x + f\cdot 5\cdot 4\cdot 3x^2 + \ldots$$

Put $x=0$; $f'''(0) = d\cdot 3! + 0 + 0$; $d = \dfrac{f'''(0)}{3!}$

Differentiate

$$f^{iv}(x) = e\cdot 4\cdot 3\cdot 2\cdot 1 + f\cdot 5\cdot 4\cdot 3\cdot 2x + \ldots$$

Put $x=0$; $f^{iv}(0) = e\cdot 4! + 0 + 0 + \ldots$; $e = \dfrac{f^{iv}(0)}{4!}$

So, $a = f(0)$, $b = f'(0)$, $c = \dfrac{f''(0)}{2!}$, $d = \dfrac{f'''(0)}{3!}$, $e = \dfrac{f^{iv}(0)}{4!}$

Putting the expression a, b, c, d, e — back to original series, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

Find the Series for $\ln(1+x)$

<u>Solution</u>

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$

$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$

$f^{iv}(x) = -3\cdot 2(1+x)^{-4} = -\frac{3\cdot 2}{(1+x)^4}$

$f^{v}(x) = 4\cdot 3\cdot 2(1+x)^{-5} = \frac{4!}{(1+x)^5}$

So we Evaluate the differentials when $x = 0$ since also $\ln 1 = 0$, substituting back into Maclaurin's series to obtain the series for $\ln(1+x)$

$f(0) = \ln 1 = 0,\ f'(0) = 1;\ f''(0) = -1,\ f'''(0) = 2$

$f^{iv}(0) = -3!,\ f^{v}(0) = 4!$

Also $f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\ln(1+x) = 0 + x\cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-3!) +$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

heads down until π radians (180°) and heads up again

The derivative of the arbitrary

$(-1)^{n-1}(n-1)!$

(K.)

Shift and Periodicity

By shifting round the arguments of trigonometric function by certain angles; it is sometimes possible that changing the sign or applying complementary trigonometric functions express particular results more simply. Some examples of shift are shown below

A full turn (360°) or 2π radian —
This does not change anything along the unit circle and makes up the smallest interval for which the trig function sin, cos, sec, and csc, repeat their values; and is thus their period. Shifting arguments of any periodic function by any integer multiple of full period preserves the function value of the unshifted argument.

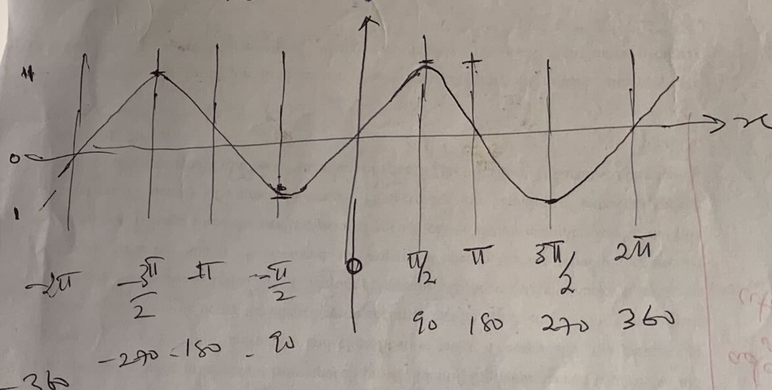
A half turn (180°) or π radian —
This the period of $\tan x = \frac{\sin(x)}{\cos(x)}$ and $\cot x = \frac{\cos(x)}{\sin(x)}$
shifting the arguments of $\tan x$ & $\cot x$ by any multiple of π, does not change their function values.

A quarter turn (90°) or $\frac{\pi}{2}$ radian —
This is a half period shift for $\tan x$ & $\cot x$ with period π (180) and yields the function value of applying the complementary function to the unshifted argument. For sin, cos, sec, & csc, a quarter turn represents a quarter period. A shift here not covered by half periods can be decomposed in an integer multiple of periods plus or minus one quarter period.

ie $= (4k \pm 1)\pi/2$

$\sin(\theta \pm \pi/2) = \pm \cos\theta$

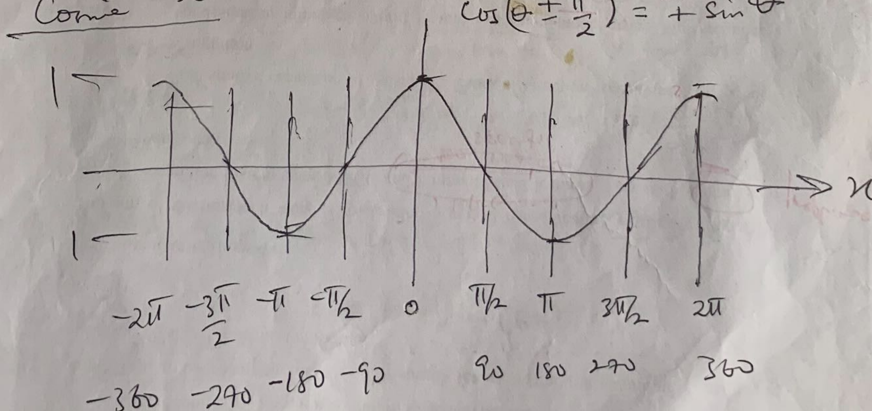
Sine x



This sine function repeats every 2π radians or $360°$: it starts at 0, heads up to 1 by $\pi/2$ radians $(90°)$ and then down to -1

Cosine x

$\cos(\theta \pm \frac{\pi}{2}) = \mp \sin\theta$



Cosine is just like sine, but it starts at 1 and heads down until π radians $(180°)$ and heads up again

(h3)

All of the other trigonometric functions can be expressed in terms of the sine and so their derivatives can easily be calculated using the rules. For cosine, we need two identities.

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\sin x = -\cos\left(x + \pi/2\right)$$

Now

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)\cdot 1 = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

Find the derivative of the following functions

(i) $\sin x \cos x$ (iii) $\sqrt{x \tan x}$ (v) $\sin(\cos(6x))$

(ii) $\sin(\cos x)$ (iv) $\sin^2 x + \cos^2 x$ (vi) $\frac{d}{dt} t^6 \cos(6t)$

i) $\cos^2 x - \sin^2 x$ (ii) $-\sin x \cos(\cos x)$ (iii) $\frac{\tan x + \sec^2 x}{2\sqrt{x \tan x}}$

iv) 0 (v) $-6\cos(\cos(6x))\sin(6x)$

vi) $5t^4\cos(6t) - 6t^6\sin(6t)$

Find all derivatives of the sine function!

<u>Solution</u>

We calculate a few derivative starting from the first one.

$$y' = (\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$y'' = (\cos x)' = -\sin x = \sin\left(x + 2\cdot\frac{\pi}{2}\right)$$

$$y''' = (-\sin x)' = -\cos x = \sin\left(x + 3\cdot\frac{\pi}{2}\right)$$

$$y^{IV} = (-\cos x)' = \sin x = \sin\left(x + 4\cdot\frac{\pi}{2}\right)$$

The n-order is expressed thus:

$$y^{n} = (\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$$

Find the nth derivative of the function $y = \sin ax$

<u>Solution</u>: Differentiating and using trig cofunction Identities

$$y' = (\sin ax)' = a\cos ax = a\sin\left(ax + \frac{\pi}{2}\right)$$

$$y'' = (a\cos ax)' = -a^2\sin ax = a^2\sin\left(ax + 2\cdot\frac{\pi}{2}\right)$$

$$y''' = (-a^2\sin ax)' = -a^3\cos ax = a^3\sin\left(ax + 3\cdot\frac{\pi}{2}\right)$$

$$y^4 = (-a^3\cos ax)' = a^4\sin ax = a^4\sin(ax + 2\pi)$$

$$= a^4\sin\left(ax + 4\cdot\frac{\pi}{2}\right)$$

The nth derivative is written in the form:

$$y^{(n)} = a^n\sin\left(ax + \frac{\pi n}{2}\right)$$

the arbitrary

(a)

$y = \sin x^3$

Solution

But $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{-} - \frac{x^7}{-} + x^9$

Find all derivatives of the cosine function:

Solution

we first find few derivatives of the cosine function

$y' = (\cos x)' = -\sin x = \cos(x + \pi/2)$

$y'' = (-\sin x)' = -\cos x = \cos(x + 2 \cdot \frac{\pi}{2})$

$y''' = (-\cos x)' = \sin x = \cos(x + 3 \cdot \pi/2)$

$y^4 = (\sin x)' = \cos x = \cos(x + 4 \cdot \pi/2)$

The nth order derivative of the function is described by $y^{(n)} = (\cos x)^{(n)} = \cos(x + \frac{n\pi}{2})$

Find the nth order derivative of the natural logarithm function $y = \ln x$.

Solution: $y' = (\ln x)' = 1/x$

$y'' = (y')' = (1/x)' = (x^{-1})' = -x^{-2} = -1/x^2$

$y''' = (y'')' = (-\frac{1}{x^2})' = 2x^{-3} = \frac{2}{x^3}$

$y^4 = (y''')' = (\frac{2}{x^3})' = -6x^{-4} = -6/x^4$

$y^5 = (y^{(4)})' = (-6/x^4)' = 24x^{-5} = 24/x^5$

The derivative of the arbitrary nth order is given by $y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$

(a)

$y = \sin x^3$

<u>Solution</u>

But $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

replacing x with x^3 -

$$\sin x^3 = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!} + \frac{(x^3)^9}{9!}$$

$$\therefore \sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!}$$

Power Series Solutions of Ordinary differe-ntial equations

① If $y = \sin x$

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = -\sin x = \sin(x+\pi) = \sin\left(x + \frac{2\pi}{2}\right)$$

Thus: $y^{(n)} = \frac{d^n y}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$

$\therefore$ If $y = \sin x$, $y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$

$y'' = -\sin x = \sin\left(x + \frac{2\pi}{2}\right)$

$y''' = -\cos x = \sin\left(x + \frac{3\pi}{2}\right)$

② If $y = \cos x$, $y' = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$

$y'' = -\cos x = \cos\left(x + \frac{2\pi}{2}\right)$

$y''' = \sin x = \cos\left(x + \frac{3\pi}{2}\right)$

thus $y^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

③ If $y = e^{ax}$, then $y' = a e^{ax}$

$y'' = a^2 e^{ax}$

$y''' = a^3 e^{ax}$

thus $y^{(n)} = a^n e^{ax}$

(4) If $y = \ln x$,

$y' = 1/x$

$y'' = -1/x^2$

$y''' = 2/x^3$

$y^4 = -3!/x^4$

$\therefore\ y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}$

Solve

(i) $y = \sin ax$, (ii) $y = \cos ax$, (iii) $y = e^{x/2}$

(iv) $y = \cos(x\sqrt{2})$

Leibnitz theorem - nth derivative of a product of two functions.

If $y = uv$, where u and v are functions of x

then: $y' = uv' + vu'$, where $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$

$y'' = uv'' + v'u' + vu'' + u'v' = u''v + 2u'v' + uv''$

$y''' = u'''v + 3u''v' + 3u'v'' + uv'''$

$y^4 = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^2v^2 + 4u^1v^3 + uv^4$

Example

Find $y^{(n)}$ when $y = x^3e^{2x}$

let $v = x^3$, $u = e^{2x}$

$u^{(n)} = 2^ne^{2x}$

ii) $y'' - xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} C_n x^n$$

The first and second derivatives of the series are given by:-

$$y'(x) = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

Inserting these derivatives into the differential equation gives:

$$0 = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} C_n n x^n - \sum_{n=0}^{\infty} C_n x^n$$

The last two sum have similar powes of x, re-indexing the first sum, let $k = n-2$, or $n = k+2$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} = \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k$$

Inserting this sum, and setting $n = k$ in the other two sums, we have

$$0 = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} C_n n x^n - \sum_{n=0}^{\infty} C_n x^n$$

$$= \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k - \sum_{k=1}^{\infty} C_k k x^k - \sum_{k=0}^{\infty} C_k x^k$$

$$= \sum_{k=1}^{\infty} [C_{k+2}(k+2)(k+1) - C_k k - C_k] x^k + C_2(2)(1) - C_0$$

$$= \sum_{k=1}^{\infty} (k+1)\left[(k+2)c_{k+2} - c_k\right]x^k + 2c_2 - c_0$$

$$2c_2 - c_0 = 0$$
$$(k+1)\left[(k+2)c_{k+2} - c_k\right] = 0 \quad , k = 1, 2, 3 \cdots$$

$$c_2 = \tfrac{1}{2}c_0$$
$$c_{k+2} = \frac{1}{k+2}c_k \quad , \quad k = 1, 2, 3 \cdots$$

Determining the coefficients with the results, thus;

$k=1$: $c_3 = \frac{1}{3}c_1$, $k=2$: $c_4 = \frac{1}{4}c_2 = \frac{1}{8}c_0$

$k=3$; $c_5 = \frac{1}{5}c_3 = \frac{1}{15}c_1$

$k=4$: $c_6 = \frac{1}{6}c_4 = \frac{1}{48}c_0$

$k=5$: $c_7 = \frac{1}{7}c_5 = \frac{1}{105}c_1$

this gives the series solution as

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$
$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$$= c_0 + c_1 x + \tfrac{1}{2}c_0 x^2 + \tfrac{1}{3}c_1 x^3 + \tfrac{1}{8}c_0 x^4 + \tfrac{1}{15}c_1 x^5$$
$$+ \tfrac{1}{48}c_0 x^6 + \cdots$$
$$= c_0\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \cdots\right) + c_1\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{115}x^5 + \cdots\right)$$

(b) Find a general Maclaurin Series Solution to the ODE : i) $y' - 2xy = 0$

Solution

Taking solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$

To find the expansion coefficients, $c_n, n = 0, 1, \cdots$

Differentiating, we have

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Inserting the series for $y(x)$ and $y'(x)$ into the differential equation, we have:

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n$$

$$= (c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \cdots)$$
$$-2x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$
$$= c_1 + (2c_2 - 2c_0)x + (3c_3 - 2c_1)x^2 + (4c_4 - 2c_2)x^3 + \cdots$$

Equating like powers of x on both sides of this result, we have

$$0 = c_1, \quad 0 = 2c_2 - 2c_0, \quad 0 = 3c_3 - 2c_1, \quad 0 = 4c_4 - 2c_2 \cdots$$

$\therefore \quad c_1 = 0, \; c_2 = c_0, \; c_3 = \frac{2}{3}c_1 = 0, \; c_4 = \frac{1}{2}c_2 = \frac{1}{2}c_0 \cdots$

$\therefore \quad y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$
$$= c_0 + c_0 x^2 + \tfrac{1}{2} c_0 x^4 + \cdots$$

2) Find the Series Solutions for the Cauchy-Euler equation :

$$ax^2y'' + bxy' + cy = 0$$

for cases i, $a=1, b=-4, c=6$

ii, $a=1, b=2, c=-6$

iii) $a=1, b=1, c=6$

Solution

$$y(x) = \sum_{n=0}^{\infty} d_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n d_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) d_n x^{n-2}$$

Inserting into the differential equation,

$$0 = ax^2y'' + bxy' + cy$$

$$= ax^2 \sum_{n=2}^{\infty} n(n-1) d_n x^{n-2} + bx \sum_{n=1}^{\infty} n d_n x^{n-1} + c\sum_{n=0}^{\infty} d_n x^n$$

$$= a\sum_{n=0}^{\infty} n(n-1) d_n x^n + b\sum_{n=0}^{\infty} n d_n x^n + c\sum_{n=0}^{\infty} d_n x^n$$

$$= \sum_{n=0}^{\infty} [an(n-1) + bn + c] d_n x^n$$

Changing the Lower limits of the first sums as $n(n-1)$ Vanishes for $n=0, 1$ and ~~setting~~ setting all coefficients to zero, we have -

$$[an^2 + (b-a)n + c] d_n = 0, \quad n = 0, 1, \ldots$$

Therefore, all of the coefficients vanish $d_n = 0$, except at the roots of $an^2 + (b-a)n + c = 0$

In the first case, $a = 1$, $b = -4$, and $c = 6$

We have

$$0 = n^2 + (-4-1)n - 6 = n^2 - 5n - 6$$
$$= (n+1)(n-6)$$

Thus, $d_n = 0$, $n \neq 6, -1$. Since the n's are nonnegative, this leaves one term in the solution

$$y(x) = d_2 x^2$$

In the second case $a = 1$, $b = 2$, and $C = -6$

We have $0 = n^2 + (2-1)n + 6 = n^2 + n + 6 =$

Since there are no real solutions to this equation

$d_n = 0$ for all n. $\cancel{y(x) = C_1 \cos}$

Finally the third case has $a = 1$, $b = 1$, and $c = 6$,

we have $0 = n^2 + (1-1)n - 6 = n^2 - 6$.

Also, there are no real solutions to this equation

$d_n = 0$ for all n.